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DISCUSSION OF HORIZONTALLY CURVED BOX BEAMS
(Published in May, 1952)

By A. George Mallis, DeForest A. Matteson, Jr., and Charles E. Cutts

STRUCTURAL DIVISION

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DISCUSSION

A. George Mallis[3], M. ASCE.—A curved profile has been a favorite design pattern of architects in the past decade, particularly in cantilevered canopies. Structural engineers have thus been faced with a problem that all too often has been completely ignored in textbooks used in undergraduate study and appears rather sparingly in graduate or advanced mechanics of materials. Practicing structural engineers have more often overcome this problem by applying empirical methods of design used in the particular design office in which they happened to be working. Mr. Cutts has done a creditable piece of work in his presentation, and his proposed method of design is certainly worthy of serious consideration for standard use in design offices.

However, as a practicing structural engineer, the writer must take serious exceptions to the author's design procedures. For example, Appendix III presents a typical design problem based on a loading of w equals 600 lb per sq ft. After a statement of the problem, the author proceeds to develop moments of 222,497.22 in.-lb, a torque of 34,338.82 in.-lb, and shear to a value of 2431.62 lb. To the writer a design analysis with computations extended to five or six significant figures, based on assumptions of two significant figures, is decidedly improper. It would have been better if the moments had been given to 223,000 in.-lb, torque to 34,300 in.-lb, and shear to 2430 lb. The same comment would apply to the determination of the required section modulus of 11.1249. The handbook of the American Institute for Steel Construction (AISC) gives section moduli to three figures, indicating that $S = 11.1$ in., cubed, would have been adequate.

DeForest A. Matteson, Jr.,[4] J.M. ASCE.—Engineers abroad used the cellular section in concrete work long before World War II, because of the high specific strength of this section in flexure and compression. Additional useful qualities have caused increasing use of the hollow girder in heavy construction in the United States. It is adapted easily to prestressed concrete construction of the precast, standardized type. Box beams are well adapted to welded steel construction, as in the author's specimens. "Locked-up" stresses in welded box beams have been studied. The flexural strength of such sections about two axes facilitates bridge construction. Box sections are valuable for resisting lateral buckling.

The torsional rigidity of the cellular section has been long realized, particularly by aircraft designers. Its value is indicated for use in girders of a straight bridge under a loading that is markedly antisymmetrical about the center line of the roadway. Straight spandrels of buildings also must resist applied torque.

Note.—This paper by Charles E. Cutts was published in May, 1952, as *Proceedings-Separate No. 128.* The numbering of footnotes, illustrations, and equations in this Separate is a continuation of the consecutive numbering used in the original paper.

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More important in this respect are the curved highway ramp, the sharply-skewed rigid frame, and the curved spandrel illustrated in the paper.

Because of the inadequate development of applicable analysis and design procedures, box sections for beams have usually been designed on a simplified basis. For concrete work, the assumptions and coefficients applicable to T-beam design have been used, giving a safe but unrealistic answer (21).[4a] The assumptions used in designing sheet metal aircraft sections may not apply to heavier structural shapes.

Commendably, the author presents a numerical example. A complete study of the horizontally curved beam must include such a test of the engineering applicability of the theory. It is the purpose of this discussion to examine some of the pitfalls that may face the designer when translating theory into a physical picture of structural behavior.

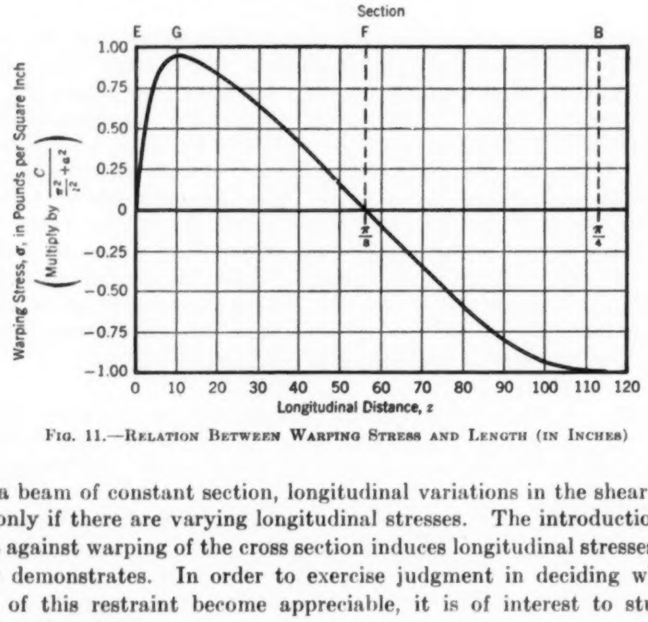

FIG. 11.—RELATION BETWEEN WARPING STRESS AND LENGTH (IN INCHES)

In a beam of constant section, longitudinal variations in the shear flow, q occur only if there are varying longitudinal stresses. The introduction of restraint against warping of the cross section induces longitudinal stresses, as the author demonstrates. In order to exercise judgment in deciding when the effects of this restraint become appreciable, it is of interest to study the variation of σ with z.

The author has shown the secondary nature (for the curved box beam) of the longitudinal stresses resisting warping. However, some useful information may be obtained from Fig. 11, which is a sketch of the variation in σ (see Eq. 16), using the constants expressed in Eqs. 17 and 18, and the physical quantities in the example of Appendix III.

The shape of the curve of σ-values in Fig. 11 is dependent on three conditions stated by the author: (a) It is assumed that the torque varies in a manner simi-

[4a] Numerals in parentheses, thus: (21), refer to corresponding items in the Bibliography (see Appendix of the paper and at the end of the discussion in this separate).

lar to the function $\sin \frac{\pi z}{l}$; (*b*) it is asserted that σ equals zero at midspan; and (*c*) it is asserted that $\sigma' = 0$ at $z = l$.

Because T varies sinusoidally—according to condition (*a*)—the stress follows a pure cosine curve except for nearly 15 in. on each side of section E. The downward deviation near E reflects the influence of the second and third terms on the right-hand side of Eq. 16. The values of these terms depend on condition (*b*). The author has not explained the basis for this condition.

The fact that there is no warping at section E is caused by symmetry of geometry and loading, and cannot form a basis for asserting that longitudinal stresses vanish at that section.

Fig. 11 shows that zero longitudinal stress does not necessarily accompany zero torque. When T is zero, σ equals either zero or σ_{max}, and when σ is zero, T equals either zero or T_{max}.

Eqs. 8 explain the situation clearly. If both the change in torsional warping and the change in torsional shear deformation are zero, σ must vanish, as at section E. If neither the warping nor the shear deformation becomes constant, but if the angle change of warping equals the negative of the angle change of shear deformation (neither of these is ϕ), σ becomes zero. Accordingly, torsional shear without restraint against torsional warping occurs at the quarter points, where T reaches its maximum value. Of course, there are additional effects caused by ordinary bending of the beam.

Therefore, it is true that σ is zero where z is zero; but the designer may also need to remember that σ reaches a peak for small values of z near section E, as shown in Fig. 11.

Because the stress σ reaches a maximum at supports A and B (see Fig. 4), condition (*c*) is reasonable. However, the symmetry of loading and geometry used by the author seems essential to his derivations. It is possible for warping of the cross section to occur at cross sections C and D. Therefore, it must be assumed that the straight parts of the beam are long and of equal stiffness, so that this warping becomes negligible in deflection computations.

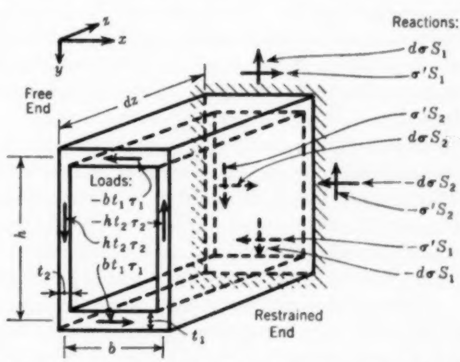

FIG. 12.—CANTILEVER BOX BEAM UNDER TORSIONAL LOAD

If span BD were removed (see Fig. 4), leaving beam CAB on three supports, the structure would be stable under the author's loading. Clearly, the value of σ at the supports depends on the ability of the straight parts to develop resistance to warping. If span BD were shorter or of smaller section (less stiff) than the rest of the beam, the σ-value at point A would exceed that at point B. The section at which σ equals zero would have shifted to one side of section

E and the solution of Eq. 16 would require the knowledge of additional conditions.

Under the author's heading, "Conclusions," number five seems to be a good suggestion, but it might cause the values of σ to be altered.

Variation in Shear Flow.—The writer believes that Eqs. 8 would be better presented if static equilibrium were more carefully indicated in Fig. 5.

Fig. 12 shows a straight cantilever box beam of length dz, free to warp at one end and perfectly restrained against warping at the other end. Shear values are positive downward and to the reader's right. Moment vectors follow the right-hand rule and are positive upward or to the right. The loads are based on the relationship:

$$q = \frac{T}{2\,b\,h} \qquad (25)$$

which gives

$$b\,t_1\,\tau_1 = \sigma'\,S_1 \qquad (26a)$$

and

$$h\,t_2\,\tau_2 = -\,\sigma'\,S_2 \qquad (26b)$$

Eqs. 26 are comparable to Eqs. 8*d* and 8*e*.

Substituting Eq. 8*c* into Eqs. 26, and solving Eqs. 26 simultaneously,

$$2\,q = \sigma'\left(\frac{2\,I_1}{b^2} - \frac{2\,I_2}{h^2}\right) \qquad (27a)$$

and

$$q = \frac{\sigma'}{12}\,(b\,t_1 - h\,t_2) \qquad (27b)$$

Eq. 25 was used by the author to determine torsional shear in Appendix III. This expression usually is derived on the assumption that longitudinal stresses are negligible (22). The value of σ' is directly proportional to the flexural shear (caused by resistance to warping) at any point along the beam.

Eq. 27*b* indicates that the q-values that would exist at a section originally free to warp are replaced by flexural shear stresses, as warping restraint is introduced. In the general case of a beam under torsion only, shear at any two sections is composed of combinations of torsional and flexural shear, the proportions varying as the warping restraint σ varies between the sections. The shear flow attributable to partial warping of the section is found by subtracting the right-hand side of Eq. 27*b* (flexural shear flow) from the right-hand side of Eq. 25 (torsional shear flow), giving Eq. 12. The author's mathematical derivation was necessary, but this physical interpretation seems helpful.

At a section having maximum restraint against torsional warping (section G, Fig. 11), the shear flow determined by Eq. 27*b* vanishes at the section and Eq. 12 reduces to Eq. 25. The opposite extreme case occurs if σ' is a maximum, in which case Eq. 12 reduces to Eq. 27*b*, even though there is no warping restraint.

Eq. 25 is often considered applicable only where there is negligible restraint. The author has made an important contribution by showing that, in the general case, Eq. 25 holds at sections on which the warping-restraint stress σ is constant in the z-direction (it may be either a maximum or a minimum), and that it is

least valid at sections where σ' is a maximum—as at section F, on which σ is zero but not constant along the beam. The case of pure torsion is thus a special (and somewhat misleading) one.

Torsional Shearing Stress.—The author's example in Appendix III has not considered the implications of Eq. 12 in the determination of the shear stress. The torsional shear is more properly determined by Eqs. 9, in which q is defined by Eq. 12.

Fig. 11 is proportional to the curve of moments produced by restraint against warping. The curve of σ'-values (the shear accompanying these moments) has its maximum value adjacent to midspan. Eq. 9*a* reduces to

$$\tau_1 = \sigma' \left(\frac{b}{\sigma} + \frac{h\, t_2}{12\, t_1} \right) \qquad (28)$$

if z equals zero. For the example in Appendix III, this yields τ_1 (torque) $= -10.3$ lb per sq in. at midspan. This value indicates that, for the beam used in the author's example, the shear effects caused by restraint against warping are negligible. Therefore, Eq. 25 is applicable as used by the author.

Under the author's heading, "Conclusions," the third conclusion is interesting when applied to the effect that warping restraint has on shear stresses. Examination of Eqs. 15 and 16 shows that the proportion suggested by the author also causes Eq. 12 to reduce to Eq. 25.

Angle of Twist.—The writer questions the importance of considering longitudinal stresses when computing ϕ. The twist per unit length ds of a hollow member that is not restrained against warping, and that is subjected to pure torque, is expressed as $\frac{T\, ds}{G\, J}$. Therefore, the angle of twist for such a member is

$$\phi = \int \frac{T\, ds}{G\, J} \qquad (29)$$

in which J is given by Eq. 6. However, the derivation of Eq. 6 is commonly based on the assumption that the longitudinal stresses equal zero everywhere in the beam (22)—a stress condition that actually occurs only at a few sections in the author's specimens.

Mr. Kuhn has warned that, for thin-walled cylinders having restraint against warping, Eq. 6 may be safely used to compute the angle of twist per unit of length only at a "considerable distance" from the section that remains plane—that is, from the section on which σ is the maximum (10).

The author's derivation of Eq. 24 for expressing ϕ is to be commended because it takes under consideration the warping stresses. Frequently, methods have been presented for analyzing curved beams, in which knowledge of a geometrical factor K (indicating the relationship between T and ϕ) has been presupposed; but the writer has seen few explanations as to how K is to be evaluated when warping stresses are present. The author indicates the complexity of the problem.

However, neglecting the effects of restraint against warping, Eq. 29 yields

$$\phi_{\max} = \int_0^{\frac{\pi}{4}} \frac{T_{\max}\, r}{G\, J} \sin \frac{\pi\, r\, \alpha}{l}\, d\alpha = - \frac{T_{\max}\, l}{G\, J\, \pi} \left(\cos \frac{\pi^2\, r}{4\, l} - 1 \right).$$

For the example in Appendix III, this equation yields ϕ_{max} (torsion) = 5.370 $\times 10^{-3}$, as compared with the value $\phi = 5.359 \times 10^{-3}$ yielded by Eq. 24. The error is approximately 0.2%, which is negligible. Thus, for this particular example, ϕ may be found by neglecting the effect of warping stresses and by using Eq. 29.

The values of both ϕ and δ are affected by the sum of the work done by the longitudinal stresses which resist warping in both the curved and straight parts of the beam. However, the work done by the beam is proportional to the sum of the squares of the various stresses. Therefore, the fact that the author's example involves a maximum value of τ (torque), which is approximately eighty times as great as the maximum σ-value, suggests that the effect of warping restraint on the twist and deflections of the curved box beam may be negligible.

Deflections.—Strictly speaking, Eq. 6 applies only to a thin-walled cylinder in torsion without end restraint. The moments $d\sigma\, S$ in Fig. 12 tend to reduce ϕ and δ. A superficial study of Eqs. 5, on the basis of the conservation of energy, indicates that to produce the same deflection value—δ at a specific z-value or s-value—a smaller applied load (therefore, less external work) will be required if there are only torsional shear stresses than if both transverse shear and longitudinal stresses are produced by the torsional loads. Therefore, the correct value of δ (torque) in the example in Appendix III should be smaller than the value given by Eq. 5 because of an increase in J based on the geometry of the section. Consideration of the energy concept indicates that knowledge of only those longitudinal stresses existing at or near the section under investigation is not a wise basis for judging the applicability of Eq. 6 in the calculation of δ.

The author has provided expressions that may be combined to form an equation expressing deflection, in which the effects of restraint against warping have been taken into account. However, in the example in Appendix III, the author has not used this mathematical tool.

The change in the angle of twist in a unit of length dz is $\phi'\, dz$, and may be expressed by substituting Eq. 21 into Eq. 20. If this angle is multiplied by the torque arm, $r \left[1 - \cos\left(\frac{\pi}{4} - \alpha\right)\right]$, and integrated between the appropriate limits, the expression for deflection will result. This expression differs from Eq. 5*b* in that the effects of warping restraint have been considered. A form of this relationship is

$$\delta\ (\text{center}) = \int_0^{\frac{\pi}{4}} \phi'\, r^2 \left[1 - \cos\left(\frac{\pi}{4} - \alpha\right)\right] d\alpha \ldots\ldots\ldots\ldots (30)$$

The right-hand side of Eq. 30, when integrated between limits, yields an unwieldy expression having more than thirty terms. The writer has not thoroughly computed ϕ by Eq. 30, but the evaluation of a few of the significant terms suggests that very little difference may exist between the author's value for δ(torque) in Appendix III and the value obtained by use of Eq. 30 in the

same example. Possibly, mathematical or numerical procedures can be devised for simplifying the evaluation of Eq. 30. Although, for large values of r, a small error in ϕ may accompany a large error in δ, the writer feels that the value yielded by Eq. 5*b* may be sufficiently accurate for most cases and that the evaluation in Appendix III is correct.

Experimental Results.—As mentioned by the author, torsion experiments have been conducted by others (see under the heading, "Introduction"). These tests have determined a constant of proportionality, K, such that $T = K\,G\,\frac{d\phi}{dz}$, for various structural sections. The constant K is a measure of torsional stiffness, which finds various applications in structural analysis. In the absence of warping restraint, $K = J$ for the closed hollow section. It might be possible for the author to study the effects of the warping stresses on this constant by tests on a curved box beam.

The author suggests that factors such as the ability of the cross section to retain its shape add greatly to the complexity of the problem. This condition indicates the importance of further testing and the possible establishment of empirical constants. The writer believes that a hollow triangular section composed of three plates, or a rectangular section to which a diagonal plate had been added, would hold its shape well, and might have some practical application.

Conclusions.—The author seems to have illustrated a fact not previously emphasized in structural engineering literature because he not only indicates that a beam that is of closed hollow section and that is continuous over the supports is highly resistant to torque (a well-known fact), but his example also suggests that for this beam the effects of restraint may be neglected in design calculations.

Neglecting secondary effects, the writer finds that the same torsion equations seem to apply to the curved box beam as those that apply to a circular tube in torsion. This fact is particularly true if $t_1/b = t_2/h$. Thus, some of the author's mathematical development becomes primarily of academic interest, and the process of design is greatly simplified.

The writer has suggested in this discussion that the expression relating shear flow and applied torque (Eq. 25) applies, regardless of the magnitude of any warping restraint existing at the section on which q is evaluated, if the restraining stresses are constant with respect to z at that section.

Charles E. Cutts,[5] A. M. ASCE.—The author agrees with Mr. Mallis regarding the number of significant places in carrying out design computations.

Mr. Matteson is to be commended for his extensive study of the paper and for his clarification of several relationships. He has indicated additional uses of the curved box beam and the comparative advantages of the box section for developing flexural and torsional strength. In regard to restraint, it should be reiterated that the supports in this problem offer no resistance to flexural or

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torsional rotation. The inner supports react vertically upward and the outer supports, vertically downward, permitting lateral as well as longitudinal rotation of the member at all supports. Structural designers are cautioned that imposing torsional restraint at any of the supports will increase the longitudinal stress produced by warping. In the experimental studies on the three types of closed sections, the reactions were transmitted to the member by steel bearing plates. Slight lateral rotation of the curved beam caused the beam to bear more on the outside section than on the inside section of the bearing plates at A and B in Fig. 4, thus creating a small torsional restraint. Because this arrangement would be used in practice, it was felt that stress studies of this nature would be more valuable than the use of ball-and-socket bearing arrangements to match the theoretical development.

Fig. 11 illustrates the variation of warping stress over the length of the beam from the center of the curved span where $z = 0$ to the inner support where $z = 113$ in. The maximum value of warping stress σ occurring 10 in. from the center of the span is only slightly less than the maximum value of warping stress σ at the support. However, the effect on design is not altered because the warping stress near the center of the span is added to a bending stress less than one half as great as the bending stress at the support. The critical design sections occur at the inner supports.

In the analysis it was assumed that $\sigma = 0$ at the center of the span at section E. However, the experimental studies indicated longitudinal warping stresses at the center of the span. The magnitude of these stresses was smaller than those at the supports A and B (23). Mr. Matteson has sought justification of this assumption by using Eqs. 8.

In comparing Eq. 26*a* and Eq. 8*d*, Mr. Matteson's value of $t_1\,\tau_1$ is only 50% of the writer's value. The writer used the relationship:

$$V_1 = \frac{dM_1}{dz} \qquad (31)$$

for the flexural shear flow, in which V_1 is the vertical shear in one side, and M_1 is the bending moment in one side. For the top or bottom side of thickness t_1 and width b, this equation becomes

$$\frac{t_1\,\tau_1\,I_1}{Q_1} = \frac{d}{dz}\left(\frac{\sigma\,I_1}{\frac{b}{2}}\right) \qquad (32)$$

in which I_1 is the moment of inertia of the side about its neutral axis perpendicular to the plane of bending. The symbol Q_1 is the statical moment of inertia for a side about the same axis and is a variable quantity. This equation reduces to Eq. 8*d*, which represents the flexural shear flow. In the "Separates" printing of the paper, confusion resulted from the erroneous substitution of S for Q in Eqs. 8. Hence, the derivation is included in this discussion. The expressions for Q are $Q_1 = \frac{t_1}{2}\left(\frac{b^2}{4} - x^2\right)$ and $Q_2 = \frac{t_2}{2}\left(\frac{h^2}{4} - y^2\right)$ in which x and y are measured from the centers of the plates to express the variation of the

flexural shearing stress. The total shear flow is equal to the sum of the flexural shear flow plus the torsional shear flow:

$$t_1 \tau_1 = \frac{d}{dz}\left[\frac{\sigma t_1}{b}\left(\frac{b^2}{4} - x^2\right)\right] + q \quad \text{(33a)}$$

$$t_2 \tau_2 = -\frac{d}{dz}\left[\frac{\sigma t_2}{h}\left(\frac{h^2}{4} - y^2\right)\right] + q \quad \text{(33b)}$$

The maximum values of the total shear flow are indicated by Eqs. 9. The variation of the flexural shear flow is parabolic across the side; thus, the flexural shear force on one side is

$$b\, t_1 \tau_1 = \tfrac{2}{3}\, b \left[\tfrac{1}{4}\frac{d}{dz}\left(\sigma\, t_1\, b\right)\right] = \frac{\sigma'\, t_1\, b^2}{6} = \sigma' S_1 \quad \text{(34)}$$

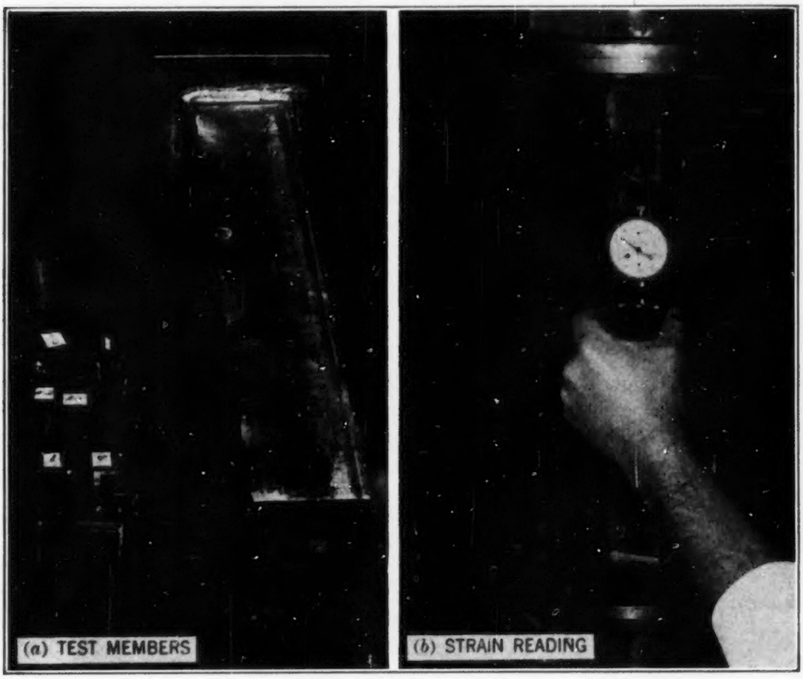

FIG. 13.—BOX BEAM TORSION TEST

Eq. 34 is identical with Mr. Matteson's Eq. 26*a* and substantiates his work. His development in expressing Eq. 25 through Eq. 27*b* checks the writer's Eq. 12 and directs attention to the two types of shear flow, namely, flexural and torsional.

With reference to Mr. Matteson's conclusions, the writer has not intended in his example to suggest that the effects of restraint may be neglected in design —rather, that longitudinal warping stresses associated with restraint may be

reduced by proper selection of box-beam proportions indicated in Conclusion 3. The writer agrees with Mr. Matteson's conclusion that Eq. 25 applies when the longitudinal warping stresses are constant. This may be shown by Eq. 12, when σ = constant, then $\sigma' = 0$ and $q = \frac{T}{2\,b\,h}$.

Mr. Matteson has suggested studies of the effect of warping stresses on the torsional stiffness constant K. In this regard, Charles R. Burke and the writer have conducted laboratory research on the torsional properties of closed box sections. Fig. 13(*a*) illustrates some of the test members composed of two structural steel angles welded together to form a box beam. Measurements of strain variations and stiffness were made on these test members. The variation of longitudinal stress in the sides of box beams has been assumed to be a straight-line relationship in this paper, as indicated in Fig. 5(*a*). With this variation, opposite corners of the beam are in the same sign of stress—either tension or compression. This means that the stress distribution is sensitive to reorientation of the axes and questions the linear variation of stress distribution. Fig. 13(*b*) shows the box beam in the torsion testing machine, in which the longitudinal strain is being measured by a 2-in. Whittemore strain gage.

Two areas of interest not touched by the discussion are the stress concentration at reentrant corners (7) and the limitations of wall thickness in the use of the thin-shell analysis methods. A projection of the present studies may be of further assistance to structural designers in determining the behavior of box beams under torsional loads.

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(22) "**Advanced Mechanics of Materials,**" by Glenn Murphy, McGraw-Hill Book Co., Inc., New York, N. Y., 1946, Chapter VII.

(23) "**Stress Analysis of Curved Box Beams Subjected to Loads Perpendicular to the Plane of Curvature,**" by Charles E. Cutts, thesis presented to the University of Minnesota, at Minneapolis, Minn., in April, 1949, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

Corrections for Transactions.—In Eq. 5*b*, add the term $d\alpha$ at the end of the expression to be integrated. In Eqs. 8*d* and 8*e*, the terms S_1 and S_2 should be changed to read Q_1 and Q_2, respectively. The line directly below Eq. 8*e* should be changed to read: "in which q denotes shear flow and Q is the statical moment of area about the neutral axis. The maximum * * *."